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Extension of Functions with ω-Rapid Polynomial Approximation

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For a weight function $\omega : [0, \infty[\to [0, \infty[$ we denote by $\mathscr{E}_{(\omega)}(\mathbb{R}^N)$ the class of all ω-ultradifferentiable functions of Beurling type on $\mathbb{R}^N$. Each element in $\mathscr{E}_{(\omega)}(\mathbb{R}^N)$ is a function with ω-rapid polynomial approximation on each compact set $K \subset \mathbb{R}^N$, whenever ω is a strong weight function, i.e.,

$$\sup_{l \in \mathbb{N}} \inf_{p \in \mathscr{P}_l^N} \|f-p\|_K e^{B\omega(l)} < \infty, \qquad \text{for all} \quad B \geqslant 1,$$

where $\mathscr{P}_l^N$ denotes the space of all polynomials in N variables of degree $\leqslant l$ and $\| \ \|_K$ denotes the sup-norm on K. In the present pages there is given a family of weight functions ω such that each function f with ω-rapid polynomial approximation defined on a compact set K satisfying Markov's inequality can be extended to an ω-ultradifferentiable function on $\mathbb{R}^N$. However this is not true for the small Gevrey classes $\mathscr{E}_{(t^{1/d})} = \Gamma^{(d)}$. © 1995 Academic Press, Inc.

By Jackson's theorem each C^∞-function f on $\mathbb{R}^N$ can be approximated on each compact set $K \subset \mathbb{R}^N$ in the following way:

$$d_{K,B}^{\infty}(f) := \sup_{l \in \mathbb{N}} \inf_{p \in \mathscr{P}_l^N} \|f-p\|_K l^B < \infty, \qquad \text{for all} \quad B \geqslant 1. \tag{1}$$

Here $\mathscr{P}_l^N$ denotes the space of all polynomials in N variables of degree $\leqslant l$ and $\| \ \|_K$ denotes the sup-norm on K. Let $s(K)$ denote the space of all continuous functions on K having property (1). Then the following restriction map is well-defined and has dense range:

$$R_K : \mathscr{E}(\mathbb{R}^N) \to s(K), \qquad R_K(f) := f|_K.$$

Here $\mathscr{E}(\mathbb{R}^N)$ denotes the space of all C^∞-functions on $\mathbb{R}^N$. Pleśniak [11] (see also [9]) has shown that the map R_K is surjective if the compact

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set K admits Markov's inequality, i.e., there exist positive numbers α and M such that

$$\|p^{(\beta)}\|_K \leqslant M l^{\alpha|\beta|} \|p\|_K, \qquad p \in \mathscr{P}_l^N, \quad l \in \mathbb{N}, \quad \beta \in \mathbb{N}_0^N. \tag{M1}$$

Moreover for a compact set $K \subset \mathbb{R}^N$ with (M1) Pleśniak constructed a continuous extension operator by explicit formulas:

$$E_K : s(K) \to \mathscr{E}(\mathbb{R}^N), \qquad E_K(f)|_K = f, \qquad f \in s(K).$$

Similar considerations for analytic functions were made by Baouendi and Goulaouic [1], Pleśniak [10], and Siciak [14].

In the present article we investigate the problem of extension of functions defined on a compact set $K \subset \mathbb{R}^N$ with certain approximation properties (which are stronger then the one in (1)) to subclasses of all C^∞-functions. For a weight function ω (see Definition 3) we denote by $\mathscr{E}_{(\omega)}$ the class of ω-ultradifferentiable functions of Beurling type, which were introduced by Beurling [2], Björck [3], and Braun *et al.* [5]. If $\omega(t) = t^{1/d}$, $d > 1$ the class $\mathscr{E}_{(\omega)}$ is equal to the small Gevrey class

$$\Gamma^{(d)}(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) \mid \text{for all } B \geqslant 1, K \subset\subset \mathbb{R}^N\text{: } \sup_{\substack{\gamma \in \mathbb{N}_0^N \\ x \in K}} |f^{(\gamma)}(x)| \frac{B^{|\gamma|}}{|\gamma|!^d} < \infty \right\}$$

(we write "$K \subset\subset \mathbb{R}^N$" when K is compact). According to Bonet *et al.* [4] the analogeue of the Whitney extension theorem holds for the class $\mathscr{E}_{(\omega)}$ if and only if ω is a strong weight function (see Remark 5(b)). Further results concerning the existence of continuous linear extension operators for ω-Whitney jets were obtained in [7] and Meise and Taylor [8]. Chaumat and Chollet [6] also investigated the problem of extension of ultradifferentiable functions for the Carleman classes $C^{\{M_p\}}$ and $C^{(M_p)}$. By results of Petzsche [12] each function $f \in \mathscr{E}_{(\omega)}(\mathbb{R}^N)$, where ω is a strong weight function, can be approximated on each set $K \subset\subset \mathbb{R}^N$ as follows:

$$d_{K,B}(f) := \sup_{l \in \mathbb{N}} \inf_{p \in \mathscr{P}_l^N} \|f - p\|_K e^{B\omega(l)} < \infty, \qquad \text{for all } \quad B \geqslant 1. \tag{2}$$

This implies as in the case of all C^∞-functions that the following map is well-defined and has dense range:

$$R_K : \mathscr{E}_{(\omega)}(\mathbb{R}^N) \to s_{(\omega)}(K), \qquad R_K(f) := f|_K.$$

Here $s_{(\omega)}(K)$ denotes the space of all continuous functions on K with property (2), endowed with the semi-norms $(d_{K,B})_{B \geqslant 1}$. The elements in $s_{(\omega)}(K)$ are called functions with ω-rapid polynomial approximation. We write $s^{(d)}(K)$ instead of $s_{(t^{1/d})}(K)$. In the following theorem we give a family of weight functions ω such that each function on K with ω-rapid polynomial approximation can be extended to an ω-ultradifferentiable function on $\mathbb{R}^N$.

THEOREM 1. *Let ω be a weight function such that for each $C > 1$ there exists $L > 1$ with $\omega(t^C) \leqslant L(\omega(t) + 1)$, $t \geqslant 0$. Suppose the compact set K admits Markov's inequality* (M1). *Then there exists a continuous extension operator*

$$E_K : s_{(\omega)}(K) \to \mathscr{E}_{(\omega)}(\mathbb{R}^N), \qquad E_K(f)|_K = f.$$

On the other hand for many weight functions ω it is impossible to extend all functions in $s_{(\omega)}(K)$ to C^∞-functions in $\mathscr{E}_{(\omega)}(\mathbb{R}^N)$.

THEOREM 2. *For each $d > 1$ the map $R_K : \Gamma^{(d)}(\mathbb{R}) \to s^{(d)}([-1, 1])$ is not surjective.*

The proofs of the theorems are based on techniques used by Pawłucki and Pleśniak [9]. To show the existence of the extension operator E_K in Theorem 1 we will use the linear topological invariant (DN), which was used by Vogt [16] to characterize the closed linear subspaces of s. In the proof of Theorem 2 we need certain estimates for the derivatives of the Chebychev polynomials.

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DEFINITION 3. For a continuous increasing function $\omega : [0, \infty[\to [0, \infty[$ we consider the following properties:

(α) there exist $K, Q > 1$ and $t_0 \geqslant 0$ such that $\omega(Kt) \leqslant Q\omega(t)$, $t \geqslant t_0$;

(β) $\displaystyle\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$;

(γ) $\displaystyle\lim_{t \to \infty} \frac{\log(t)}{\omega(t)} = 0$;

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex;

(ε) there exists $C > 0$ such that $\displaystyle\int_1^\infty \frac{\omega(yt)}{t^2}\, dt \leqslant C(\omega(y) + 1)$, for all $y \geqslant 0$.

The function ω is called a weight function (strong weight function) if it satisfies (α), (β), (γ), (δ) (and (ε)). For $x \leqslant 0$ we set $\omega(x) := \omega(-x)$. The Young conjugate $\varphi_\omega^* : [0, \infty[\to \mathbb{R}$ of φ_ω is defined by

$$\varphi_\omega^*(y) := \sup\{xy - \varphi_\omega(x) \mid x \geqslant 0\}, \qquad y \geqslant 0.$$

DEFINITION 4. Let ω be a weight function.

(a) Let $\Omega \subset \mathbb{R}^N$ be open. For $f \in C^\infty(\Omega)$, $K \subset\subset \Omega$, and $l \in \mathbb{N}$ we define:

$$\|f\|_{K,l} := \sup_{|\alpha| \leqslant l} \|f^{(\alpha)}\|_K,$$

where $\|\ \|_K$ denotes the sup-norm on K. Moreover we define the space

$$\mathscr{E}_{(\omega)}(\Omega) := \Big\{ f \in C^\infty(\Omega) \mid \text{for all } K \subset\subset \Omega,\ B \geqslant 1:$$

$$\|f\|^{\omega}_{K,B} := \sup_{l \in \mathbb{N}} \|f\|_{K,l} \exp\left(-B\varphi^*_\omega\left(\frac{l}{B}\right)\right) < \infty \Big\},$$

endowed with its natural projective limit topology.

(b) Let $A \subset \mathbb{R}^N$ be closed. For a Whitney jet $f \in \mathscr{E}(A)$, $K \subset\subset A$, and $l \in \mathbb{N}$ we set

$$|f|_{K,l} := \sup_{\substack{x, y \in K \\ x \neq y}} \sup_{|\alpha| \leqslant l} \frac{|(R^l_x f)^\alpha(y)|}{|x-y|^{l+1-|\alpha|}} (l+1-|\alpha|)! + \|f\|_{K,l},$$

where

$$(R^l_x f)^\alpha(y) := f^{(\alpha)}(y) - \sum_{|\beta| \leqslant l-|\alpha|} \frac{1}{\beta!} f^{\alpha+\beta}(x)(y-x)^\beta.$$

We define the space

$$\mathscr{E}_{(\omega)}(A) := \Big\{ f \in \mathscr{E}(A) \mid \text{for all } K \subset\subset A,\ B \geqslant 1:$$

$$|f|^{\omega}_{K,B} := \sup_{l \in \mathbb{N}} |f|_{K,l} \exp\left(-B\varphi^*_\omega\left(\frac{l+1}{B}\right)\right) < \infty \Big\},$$

endowed with the projective limit topology.

The elements of $\mathscr{E}_{(\omega)}(\Omega)(\mathscr{E}_{(\omega)}(A))$ *are called* ω-ultradifferentiable functions (Whitney jets) of Beurling type on Ω (on A).

Remark 5. (a) For a weight function ω Braun *et al.* [5] proved that there exist non-trivial functions in $\mathscr{E}_{(\omega)}(\mathbb{R}^N)$ with compact support, i.e., the classes $\mathscr{E}_{(\omega)}$ are non-quasianalytic. $\mathscr{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space and $\mathscr{E}_{(\omega)}(A)$ is a Fréchet–Schwartz space.

(b) According to Bonet *et al.* [4] for each closed set $A \subset \mathbb{R}^N$ and each ω-Whitney jet $f \in \mathscr{E}_{(\omega)}(A)$ there exists a function $F \in \mathscr{E}_{(\omega)}(\mathbb{R}^N)$ such that $F^{(\alpha)}|_A = f^\alpha$, $\alpha \in \mathbb{N}_0^N$ if and only if ω is a strong weight function.

(c) The following functions are weight functions:

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$

(2) $\omega(t) = (\log(1+t))^\beta$, $\beta > 1$

(3) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.

(4) $\omega(t) = \exp(\beta(\log(1+t))^\alpha)$, $0 < \alpha < 1$, $\beta > 0$.

The weight functions in (1), (2), and (4) are strong weight functions.

DEFINITION 6. (a) Let $\mathscr{P}_l^N$ denote the set of all polynomials of degree at most l. Set $\mathscr{P}_{-1}^N := \{0\}$. For a closed set $A \subset \mathbb{R}^N$ we denote by $C(A)$ the space of all continuous functions on A. Given $f \in C(A)$ and a compact set $K \subset A$ we define for $j \in \mathbb{N}_0 \cup \{-1\}$

$$d_K(f, \mathscr{P}_l^N) := \inf\{\|f - p\|_K \mid p \in \mathscr{P}_l^N\}.$$

(b) Let ω be a weight function and $A \subset \mathbb{R}^N$ be a closed set. We define the space

$$s_{(\omega)}(A) = \{f \in C(A) \mid \text{for each } B \geq 1, K \subset\subset A:$$
$$d_{K,B}(f) := \sup_{l \geq -1} d_K(f, \mathscr{P}_l^N)\, e^{B\omega(l)} < \infty\},$$

where $s_{(\omega)}(A)$ endowed with the semi-norms $(d_{K,B})_{K \subset\subset A, B \geq 1}$ is a Fréchet space. The elements of $s_{(\omega)}(A)$ are called functions with ω-rapid polynomial approximation. We write $s^{(d)}(A)$ instead of $s_{(t^{1/d})}(A)$.

Remark 7. Let ω be a strong weight function. From Petzsche [12] we have the following topological identity: $\mathscr{E}_{(\omega)}(\mathbb{R}^N) = s_{(\omega)}(\mathbb{R}^N)$. One can check that for each closed set $A \subset \mathbb{R}^N$ the map

$$R_{\omega,A} : \mathscr{E}_{(\omega)}(A) \to s_{(\omega)}(A), \qquad R_{\omega,A}((f^\alpha)_{\alpha \in \mathbb{N}_0^N}) := f^0$$

is well-defined, linear, and continuous. The map $R_{\omega,A}$ is injective if A is a C^∞-determining set; i.e., for each $f \in \mathscr{E}(A)$ the property $f^0|_A = 0$ implies $f^\alpha|_A = 0$, for each $\alpha \in \mathbb{N}_0^N$. In this case we identify $\mathscr{E}_{(\omega)}(A)$ with a subset of $s_{(\omega)}(A)$.

PROPOSITION 8. *Let ω be a strong weight function and $K \subset \mathbb{R}^N$ be a compact, C^∞-determining set. Then the following are equivalent:*

(1) $\mathscr{E}_{(\omega)}(K) = s_{(\omega)}(K)$;

(2) *for each $B \geq 1$ there exist $D, C > 0$ such that for all $l \in \mathbb{N}$, $p \in \mathscr{P}_l^N$, and $m \in \mathbb{N}$ we have*

$$|p|_{K,m} \leq C \exp\left(B\varphi_\omega^*\left(\frac{m+1}{B}\right) + D\omega(l)\right) \|p\|_K.$$

Proof. (1)$\Rightarrow$(2). By the open mapping theorem $R_{\omega,K}$ is a topological isomorphism. Thus for each $B\geqslant 1$ there exist $D_1\geqslant 1$, $C_1>0$ so that for all $f\in s_{(\omega)}(K)$ the following holds:

$$|R_{\omega,K}^{-1}(f)|_{K,B}^{\omega}\leqslant C_1\, d_{K,D_1}(f).$$

This implies for $p\in\mathscr{P}_l^N\subset s_{(\omega)}(K)$

$$\begin{aligned}&\sup_{m\in\mathbb{N}_0}|p|_{K,m}\exp\left(-B\varphi_\omega^*\left(\frac{m+1}{B}\right)\right)\\&\quad\leqslant C_1\sup_{l'\geqslant -1}\operatorname{dist}(p,\mathscr{P}_{l'}^N)\,e^{D_1\omega(l')}\\&\quad\leqslant C_1\sup_{l\geqslant l'\geqslant -1}\inf_{q\in\mathscr{P}_{l'}^N}\|p-q\|_K\,e^{D_1\omega(l')}\\&\quad\leqslant C_1\sup_{l\geqslant l'\geqslant -1}\|p\|_K\,e^{D_1\omega(l')}\leqslant\|p\|_K\,e^{D_1\omega(l)}.\end{aligned}$$

Hence we have shown (2).

(2)$\Rightarrow$(1). Let $f\in s_{(\omega)}(K)$. For each $l\in\mathbb{N}$ there exists a polynomial $p_l\in\mathscr{P}_l^N$ such that $\operatorname{dist}_K(f,\mathscr{P}_l^N)=\|f-p_l\|_K$. Since ω satisfies condition (α) there exists $L>1$ such that $\omega(t+1)\leqslant L\omega(t)+L$, $t\geqslant 0$. Let $B\geqslant 1$ be arbitrarily given. Choose numbers $D\geqslant 1$ and $C>0$ as in (2). We get for $m,n,l\in\mathbb{N}$

$$\begin{aligned}&\left|\sum_{j=1}^{n}(p_j-p_{j-1})\right|_{K,m}\\&\quad\leqslant\sum_{j=1}^{n}|p_j-p_{j-1}|_{K,m}\\&\quad\leqslant C\sum_{j=1}^{n}\|p_j-p_{j-1}\|_K\exp\left(B\varphi_\omega^*\left(\frac{m+1}{B}\right)+D\omega(j)\right)\\&\quad\leqslant C\exp\left(B\varphi_\omega^*\left(\frac{m+1}{B}\right)\right)\sum_{j=1}^{n}(\|p_j-f\|_K+\|p_{j-1}-f\|_K)\,e^{D\omega(j)}\\&\quad\leqslant 2Ce^{DL}\exp\left(B\varphi_\omega^*\left(\frac{m+1}{B}\right)\right)\left(\sum_{j=1}^{\infty}e^{-\omega(j)}\right)d_{K,L(D+1)}(f).\end{aligned}$$

Hence the sequence $(p_j)_{j\in\mathbb{N}}=(p_0-\sum_{j=1}^{n}(p_j-p_{j-1}))_{n\in\mathbb{N}}$ converges to a function g in $\mathscr{E}_{(\omega)}(K)$. It is easy to check that $R_{\omega,K}(g)=f$.

From Vogt [16] we recall the following linear topological invariant.

DEFINITION 9. Let F be a Fréchet space and $\mathscr{P}:=(p_j)_{j\in\mathbb{N}}$ be a fundamental system of semi-norms of F. F is said to have the property (DN) if there exists $m\in\mathbb{N}$ such that for each $k\in\mathbb{N}$ there exist numbers $l\in\mathbb{N}$ and $C>0$ so that $\| \; \|_k^2\leqslant C\| \; \|_m \| \; \|_l$.

Proof of Theorem 1. First we will show that $\mathscr{E}_{(\omega)}(K)=s_{(\omega)}(K)$. Since K is C^∞-determining by [11, 3.5] it suffices to show that Proposition 8(2) is satisfied. The proof of [11, 3.3] implies that there exist numbers $M>0$ and $A\geqslant 1$ such that for all $l, m\in\mathbb{N}$ and $p\in\mathscr{P}_l^N$,

$$|p|_{K,m}\leqslant Ml^{Am}\|p\|_K.$$

Choose numbers $C>0$ and $t_0\geqslant 0$ such that $\omega(t^A)\leqslant C\omega(t)$, $t\geqslant t_0$. It is no loss of generality to take $t_0=0$. For all $l, m\in\mathbb{N}$ and $p\in\mathscr{P}_l^N$ the following holds:

$$\begin{aligned}|p|_{K,m}&\leqslant Ml^{Am}\|p\|_K=M\exp(m\log(l^A)-\omega(l)+\omega(l))\|p\|_K\\ &\leqslant M\exp(\sup_{l'\geqslant 0}(m\log(l'^A)-B\omega(l'))+B\omega(l))\|p\|_K\\ &\leqslant M\exp(\sup_{l'\geqslant 0}(ml'-B\omega(e^{l'/A}))+B\omega(l))\|p\|_K\\ &\leqslant M\exp\left(\sup_{l'\geqslant 0}\left(ml'-\frac{B}{C}\omega(e^{l'})\right)+B\omega(l)\right)\|p\|_K\\ &=M\exp\left(\frac{B}{C}\varphi_\omega^*\left(\frac{m}{B/C}\right)+B\omega(l)\right)\|p\|_K.\end{aligned}$$

Then $8(2)\Rightarrow(1)$ implies $\mathscr{E}_{(\omega)}(K)=s_{(\omega)}(K)$. By [7, 4.8] there exists a continuous linear extension operator on K if and only if the space $\mathscr{E}_{(\omega)}(K)$ has the property (DN). Since the topology on $s_{(\omega)}(K)$ coincides with the topology on $\mathscr{E}_{(\omega)}(K)$ it suffices to show that the space $s_{(\omega)}(K)$ has (DN). In doing so let $B\geqslant 1$ be arbitrarily given. For each $f\in s_{(\omega)}(K)$ we have

$$\begin{aligned}d_{K,B}^2(f)&=\sup_{l\geqslant -1} d_K(f,\mathscr{P}_l^N)^2 e^{2B\omega(l)}\\ &\leqslant(\sup_{l\geqslant -1} d_K(f,\mathscr{P}_l^N)e^{\omega(l)})(\sup_{l\geqslant -1} d_K(f,\mathscr{P}_l^N)e^{(2B-1)\omega(l)})\\ &=d_{K,1}(f)\,d_{K,2B-1}(f).\end{aligned}$$

Thus the proof is complete.

Remark 10. It is easy to see that the functions $\omega_s(t)=[\log^+(t)]^s$, $s>1$ satisfy the properties in Theorem 1. On the other hand if ω satisfies the hypothesis of Theorem 1 then there exists a number $s>0$ such that $\omega(t)=O([\log^+(t)]^s)$, $t\to\infty$. The function $\omega_1=\log^+$ is not a weight function since it does not satisfy condition (γ). Nevertheless we identify $\mathscr{E}_{(\omega)}(\mathbb{R}^N)$ with the set of all C^∞-functions. In this case Theorem 1 has already been proved by Pawłucki and Pleśniak [9].

Proof of Theorem 2. By Remarks 5(b) and 5(c) the map

$$R_{[-1,1]}: \Gamma^{(d)}(\mathbb{R}) \to \Gamma_{(d)}([-1,1]), \qquad R_{[-1,1]}(f) = f|_{[-1,1]}$$

is surjective. To prove Theorem 2 it therefore suffices to show that 8(2) does not hold. It is easy to check that $\varphi_\omega^*(x) = d \cdot x \log(dx/e)$, $x \geqslant e$. Hence we have to prove the following:

(1) There exists $B_0 \geqslant 1$ such that for all $D \geqslant 1$ and $C > 0$ there exist numbers $r, n \in \mathbb{N}$ and $p \in \mathscr{P}_n^1$ with

$$\|p^{(r)}\|_{[-1,1]} \geqslant \exp\left(r \log\left(\frac{r^d}{B_0}\right) + Dn^{1/d} + C\right) \|p\|_{[-1,1]}.$$

To show (1) we will use the Chebychev polynomials

$$T_n(x) = \cos(n \arccos(x)), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

T_n is a polynomial of degree n. Put $r_0 = (e-2)^{-1}$. We first show that for each $n \in \mathbb{N}$ and $r \in \mathbb{N}$, $r_0 \leqslant r \leqslant n-1$ the following holds:

$$(2) \quad T_n^{(r)}(1) \geqslant \exp\left(n \log\left(\frac{n+r}{n-r}\right) + r \log\left(\frac{n^2-r^2}{e^2 r}\right) + \frac{1}{2}\log(er)\right).$$

By Timan [15, p. 226] for each $n \in \mathbb{N}$ and $r \in \mathbb{N}$, $r \leqslant n-1$ we have

$$\begin{aligned}
T_n^{(r)}(1) &= \frac{n^2(n^2-1)\cdot \cdots \cdot (n^2-(r-1)^2)}{1 \cdot 3 \cdot \cdots \cdot (2r-1)} \\
&= \exp\left(\sum_{j=0}^{r-1} \log(n^2-j^2) - \sum_{j=1}^{r} \log(2j-1)\right) \\
&\geqslant \exp\left(\int_0^r \log(n^2-j^2)\, dj - \int_1^{r+1} \log(2j-1)\, dj\right) \\
&= \exp\left(-(n-j)\log\left(\frac{n-j}{e}\right)\Big|_0^r \right.\\
&\quad \left. + (n+j)\log\left(\frac{n+j}{e}\right)\Big|_0^r - \frac{1}{2}(2j-1)\log\left(\frac{2j-1}{e}\right)\Big|_1^{r+1}\right) \\
&= \exp\left((n+r)\log\left(\frac{n+r}{e}\right)\right. \\
&\quad \left. -(n-r)\log\left(\frac{n-r}{e}\right) - \left(r+\frac{1}{2}\right)\log\left(\frac{2r+1}{e}\right) + \frac{1}{2}\right).
\end{aligned}$$

With $r \geqslant r_0$ we obtain

$$T_n^{(r)}(1) \geqslant \exp\left((n+r)\log\left(\frac{n+r}{e}\right) - (n-r)\log\left(\frac{n-r}{e}\right) - r\log(r) - \frac{1}{2}\log(er)\right)$$
$$= \exp\left(n\log\left(\frac{n+r}{n-r}\right) + r\log\left(\frac{n^2-r^2}{e^2}\right) - r\log(r) - \frac{1}{2}\log(er)\right)$$
$$= \exp\left(n\log\left(\frac{n+r}{n-r}\right) + r\log\left(\frac{n^2-r^2}{e^2 r}\right) - \frac{1}{2}\log(er)\right).$$

Hence (2) is shown. Now choose a number $B_0 \geqslant e^2$. Let $C, D > 0$ be arbitrarily given. Then for all $n, r \in \mathbb{N}$, $r_0 \leqslant r \leqslant n-1$ the following holds:

$$\|T_n^{(r)}\|_{[-1,1]} \exp\left(-r\log\left(\frac{r^d}{B_0}\right) - Dn^{1/d} - C\right)$$
$$\geqslant T_n^{(r)}(1) \exp\left(-r\log\left(\frac{r^d}{B_0}\right) - Dn^{1/d} - C\right)$$
$$\geqslant \exp\left(n\log\left(\frac{n+r}{n-r}\right) + r\log\left(\frac{n^2-r^2}{re^2}\right) - r\log\left(\frac{r^d}{B_0}\right) - Dn^{1/d} - \frac{1}{2}\log(er) - C\right)$$
$$= \exp\left(n\log\left(\frac{n+r}{n-r}\right) + r\log\left(\frac{n^2-r^2}{r^{d+1}}\frac{B_0}{e^2}\right) - Dn^{1/d} - \frac{1}{2}\log(er) - C\right).$$

This implies for $r \in \mathbb{N}$, $r \geqslant r_0$ with $n := r^{(d+1)/2} \in \mathbb{N}$:

$$\|T_n^{(r)}\|_{[-1,1]} \exp\left(-r\log\left(\frac{r^d}{B_0}\right) - Dn^{1/d} - C\right)$$
$$\geqslant \exp\left(r^{(d+1)/2}\log\left(\frac{r^{(d+1)/2}+r}{r^{(d+1)/2}-r}\right) + r\log\left(\frac{r^{d+1}-r^2}{r^{d+1}}\frac{B_0}{e^2}\right) - Dr^{(d+1)/2d} - \frac{1}{2}\log(er) - C\right)$$

$$= \exp\left(r^{(d+1)/2} \log\left(\frac{r^{(d-1)/2}+1}{r^{(d-1)/2}-1}\right)\right.$$

$$\left. + r \log\left(\frac{r^{d+1}-r^2}{r^{d+1}} \frac{B_0}{e^2}\right) - Dr^{(d+1)/2d} - \frac{1}{2}\log(er) - C\right)$$

$$\geqslant \exp\left(r^{d+1/2} \log\left(1 + \frac{2}{r^{d-1/2}-1}\right)\right.$$

$$\left. + r \log\left(1 - \frac{1}{r^{d-1}}\right) - Dr^{(d+1)/2d} - \frac{1}{2}\log(er) - C\right)$$

$$\geqslant \exp\left(r^{(d+1)/2} \frac{\frac{2}{r^{(d-1)/2}-1}}{\frac{2}{r^{(d-1)/2}-1}+1} + r \frac{-\frac{1}{r^{d-1}}}{\frac{1}{r^{d-1}}+1} - Dr^{(d+1)/2d} + \frac{1}{2}\log(er) - C\right)$$

$$= \exp\left(2r \frac{r^{(d-1)/2}}{r^{(d-1)/2}+1} + \frac{1}{2} r \frac{\log(er)}{r}\right.$$

$$\left. - r \frac{1}{r^{d-1}-1} - Dr \frac{1}{r^{(d+2)/(2d+2)}} - r \frac{C}{r}\right)$$

$$=: \exp(rA(D, C, r)).$$

Obviously one can find a number $r \in \mathbb{N}$ with $n := r^{(d+1)/2} \in \mathbb{N}$ such that $A(D, C, r) \geqslant 0$. We get the following inequality:

$$\|T_n^{(r)}\|_{[-1,1]} \exp\left(-r \log\left(\frac{r^d}{B_0}\right) - Dn^{1/d} - C\right) \geqslant 1 = \|T_n\|_{[-1,1]}.$$

Hence we have shown (1).

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